

QUANTUM MIRROR CURVES FOR \mathbb{C}^3 AND THE RESOLVED CONIFOLD

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ABSTRACT. We establish a conjecture of Gukov and Sułkowski in the following three cases: Lambert curve for Hurwitz numbers, framed mirror curve of \mathbb{C}^3 , and the framed mirror curve of the resolved conifold.

Key words. Open string invariants, Eynard-Orantin topological recursion, quantum mirror curves.

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In this sequel to [23] we treat three more cases of a conjecture made by Gukov and Sułkowski [12]. We will study the following open string invariants: Hurwitz numbers, one-legged topological vertex, and the resolved conifold with one framed outer D-brane. By the BKMP remodeling conjecture [16, 3, 4], in each of these cases, the open string invariants can be encoded in some curve on which one can define some differentials by the Eynard-Orantin recursion [9]. For the proofs in these cases, see [2, 8, 5, 20, 10]. In each case the mirror curve are given by an equation

$$A(u, v) = 0, \quad u, v \in \mathbb{C}$$

or

$$A(x, y) = 0, \quad x, y \in \mathbb{C}^*.$$

Gukov and Sułkowski [12] defined some partition function Z and conjectured that there is a quantization $\hat{A}(\hat{x}, \hat{y})$ of $A(x, y)$ into differential operator such that

$$(1) \quad \hat{A}(\hat{x}, \hat{y})Z = 0.$$

In [23] we have established this in the case of the Airy curve. In this paper we will deal with the three cases mentioned above. We will use the known results of the corresponding A-model calculations in each case, and derivation of the quantum mirror curves follows that of [12, (6.7)-(6.12)].

1. HURWITZ NUMBERS AND QUANTUM LAMBERT CURVE

1.1. Hurwitz numbers and Burnside formula. For a partition $\mu = (\mu_1, \dots, \mu_{l(\mu)})$ of $d > 0$, denote by $H_{g,\mu}$ the Hurwitz number of branched coverings of \mathbb{P}^1 of type μ by genus g Riemann surfaces. In general, these numbers can be computed by the Burnside formula:

$$(2) \quad \exp \sum_{\mu \neq \emptyset} \sum_{g \geq 0} \frac{\lambda^{2g-2+l(\mu)+|\mu|}}{(2g-2+l(\mu)+|\mu|)!} H_{g,\mu} p_\mu = \sum_{\nu} \frac{\dim R_\nu}{|\nu|!} \cdot e^{\kappa_\nu \lambda/2} \cdot s_\nu.$$

Here $p_\mu = \prod_{i=1}^{l(\mu)} p_{\mu_i}$ are the Newton functions, s_ν are the Schur functions. They are related to each other by the characters of irreducible representations of the symmetric groups:

$$(3) \quad s_\mu = \sum_{\nu} \frac{\chi_\mu(\nu)}{z_\nu} p_\nu, \quad p_\nu = \sum_{\mu} \chi_\mu(\nu) s_\mu,$$

where χ_μ denotes the character of the irreducible representation R_μ indexed by μ and $\chi_\mu(\nu)$ denotes its value on the conjugacy class indexed by ν . For a partition $\mu = (\mu_1, \dots, \mu_l)$, the number κ_μ is defined as follows:

$$(4) \quad \kappa_\mu = \sum_{i=1}^l \mu_i (\mu_i - 2i + 1).$$

1.2. Hurwitz numbers and the cut-and-join equation. Denote by H^\bullet the left-hand side of (2). It satisfies the following differential equation, called the *cut-and-join equation* [11]:

$$(5) \quad \frac{\partial H^\bullet}{\partial \lambda} = K H^\bullet,$$

where

$$(6) \quad K = \frac{1}{2} \sum_{i,j \geq 1} \left(i j p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right).$$

This is because

$$(7) \quad K s_\nu = \frac{1}{2} \kappa_\nu \cdot s_\nu.$$

1.3. Hurwitz numbers and ELSV formula. Hurwitz numbers are related to Hodge integrals on the Deligne-Mumford moduli spaces by the ELSV formula [7]:

$$(8) \quad H_{g,\mu} = \frac{1}{|\text{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)},$$

where $\Lambda_g^\vee(1) = \sum_{i=0}^g (-1)^i \lambda_i$.

1.4. Symmetrization. One can also define

$$H_g(\mathbf{p}) = \sum_{\mu} \frac{1}{(2g-2+l(\mu)+|\mu|)!} H_{g,\mu} p_{\mu}.$$

Because $H_g(p)$ is a formal power series in $p_1, p_2, \dots, p_n, \dots$, for each n , one can obtain from it a formal power series $\Phi_{g,n}(x_1, \dots, x_n)$ by applying the following linear symmetrization operator [11]:

$$p_{\mu} \mapsto \delta_{l(\mu),n} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}.$$

Because

$$\sum_{\nu} e^{\kappa_{\nu} \lambda/2} \cdot \frac{\dim R_{\nu}}{|\nu|!} \cdot s_{\nu} = 1 + \sum_{|\nu| \geq 1} e^{\kappa_{\nu} \lambda/2} \frac{\dim R_{\nu}}{|\nu|!} \sum_{\mu} \frac{\chi_{\nu}(\mu)}{z_{\mu}} p_{\mu},$$

we have

$$\begin{aligned} & \log \sum_{\nu} e^{\kappa_{\nu} \lambda/2} \frac{\dim R_{\nu}}{|\nu|!} \cdot s_{\nu} \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{\mu^1 \cup \dots \cup \mu^k = \mu \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} \sum_{|\nu^i| = |\mu^i|} e^{\kappa_{\nu^i} \lambda/2} \frac{\dim R_{\nu^i}}{|\nu^i|!} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \cdot p_{\mu}. \end{aligned}$$

It follows that

$$\begin{aligned} & \sum_{g \geq 0} \frac{\lambda^{2g-2+l(\mu)+|\mu|}}{(2g-2+l(\mu)+|\mu|)!} H_{g,\mu} p_{\mu} \\ (9) \quad &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{\mu^1 \cup \dots \cup \mu^k = \mu \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{\kappa_{\nu^i} \lambda/2} \frac{\dim R_{\nu^i}}{|\nu^i|!} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \cdot p_{\mu}. \end{aligned}$$

After the symmetrization,

$$\begin{aligned} & \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x_1, \dots, x_n) \\ (10) \quad &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{l(\mu^1) + \dots + l(\mu^k) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{\kappa_{\nu^i} \lambda/2} \frac{\dim R_{\nu^i}}{|\nu^i|!} \cdot \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \\ & \cdot \lambda^{-|\mu|} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}, \end{aligned}$$

where $\mu = (\mu_1, \dots, \mu_n) = \mu^1 \cup \dots \cup \mu^k$. In particular,

$$(11) \quad \begin{aligned} & \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x) \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{\sum_{i=1}^k l(\mu^i) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{\kappa_{\nu^i} \lambda/2} \frac{\dim R_{\nu^i}}{|\nu^i|!} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \left(\frac{x}{\lambda}\right)^{|\mu^i|}. \end{aligned}$$

1.5. Computation of a partition function.

Proposition 1.1. *Let*

$$(12) \quad Z = \exp \sum_{n \geq 1} \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x).$$

Then one has

$$(13) \quad Z = \sum_{n=0}^{\infty} e^{n(n-1)\lambda/2} \frac{x^n}{n! \lambda^n}.$$

Proof. By (11) we have

$$\begin{aligned} Z &= 1 + \sum_{|\mu| > 0} \sum_{|\nu| = |\mu|} e^{\kappa_{\nu} \lambda/2} \frac{\dim R_{\nu}}{|\nu|!} \frac{\chi_{\nu}(\mu)}{z_{\mu}} \cdot (x/\lambda)^{|\mu|} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu| = n} (x/\lambda)^{|\nu|} e^{\kappa_{\nu} \lambda/2} \frac{\dim R_{\nu}}{|\nu|!} \sum_{|\mu| = n} \frac{\chi_{\nu}(\mu)}{z_{\mu}} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu| = n} (x/\lambda)^{|\nu|} e^{\kappa_{\nu} \lambda/2} \frac{\dim R_{\nu}}{|\nu|!} \delta_{\nu, (n)} \\ &= 1 + \sum_{n=1}^{\infty} (x/\lambda)^n e^{\kappa_{(n)} \lambda/2} \frac{\dim R_{(n)}}{n!} \\ &= \sum_{n=0}^{\infty} e^{n(n-1)\lambda/2} \frac{x^n}{n! \lambda^n}. \end{aligned}$$

□

1.6. Differential equation satisfied by the partition function.

Write $Z = a_0 + a_1 + \dots$, where

$$(14) \quad a_n = e^{n(n-1)\lambda/2} \frac{x^n}{n! \lambda^n}.$$

Then one has

$$(15) \quad \frac{a_{n+1}}{a_n} = \frac{x}{(n+1)\lambda} e^{n\lambda},$$

and so

$$(16) \quad (n+1)\lambda a_{n+1} - x e^{n\lambda} a_n = 0.$$

By summing over n , one gets:

Theorem 1.2. *For the partition function associated with the Hurwitz numbers define above, the following equation is satisfied:*

$$(17) \quad (\hat{y} - \hat{x}e^{\hat{y}})Z = 0,$$

where

$$(18) \quad \hat{x} = x \cdot, \quad \hat{y} = \lambda x \frac{\partial}{\partial x}.$$

This established the Lambert curve case of Gukov-Sulkowski conjecture [12]. Recall that Bouchard and Mariño [4] conjectured that

$$(19) \quad W_{g,n}(x_1, \dots, x_n) = \partial_{x_1} \cdots \partial_{x_n} \Phi_{g,n}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

satisfies the Eynard-Orantin recursion defined by the Lambert curve:

$$(20) \quad A(x, y) = y - x e^y = 0.$$

This is sort of a limiting case of the \mathbb{C}^3 case of the BKMP conjecture [3]. This conjecture has been proved by Borot-Eynard-Safnuk-Mulase [2] and Eynard-Safnuk-Mulase [8] by two different methods. Our result shows that the quantization of $A(x, y)$ does not involve higher order quantum corrections in this case.

2. MARIÑO-VAFA FORMULA AND QUANTUM MIRROR CURVE OF \mathbb{C}^3

2.1. Mariño-Vafa formula. For a partition $\mu = (\mu_1, \dots, \mu_{l(\mu)})$, consider the triple Hodge integral:

$$\begin{aligned} \mathcal{C}_{g,\mu}(a) = & -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\text{Aut}(\mu)|} (a(a+1))^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1} (\mu_i a + a)}{(\mu_i - 1)!} \\ & \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-a-1) \Lambda_g^\vee(a)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)}. \end{aligned}$$

Note when $l(\mu) \geq 3$, we have

$$(21) \quad \int_{0,l(\mu)} \frac{\Lambda_0^\vee(1) \Lambda_0^\vee(a) \Lambda_0^\vee(-1-a)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = \int_{0,l(\mu)} \frac{1}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = |\mu|^{l(\mu)-3},$$

We use this to extend the definition to the case of $(g, n) = (0, 1)$ and $(0, 2)$. The Mariño-Vafa formula [17, 13, 18] states that

$$(22) \quad \sum_{|\mu| \geq 1} \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(a) p_\mu = \log \sum_{|\nu| \geq 0} q^{a\kappa_\nu \tau/2} \sqrt{-1}^{|\nu|} s_\nu(q^\rho) s_\nu,$$

where $q = e^{\sqrt{-1}\lambda}$, $s_\nu(q^\rho) = s_\nu(q^{-1/2}, q^{-3/2}, \dots)$. Equivalently,

$$(23) \quad \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(a) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{\mu^1 \cup \dots \cup \mu^k = \mu \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} q^{a\kappa_{\nu^i}/2} \sqrt{-1}^{|\nu^i|} s_{\nu^i}(q^\rho) \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}}.$$

2.2. Symmetrization. For fixed $g \geq 0$ and $n \geq 1$, define

$$\mathcal{C}_{g,n}(p; a) = \sum_{l(\mu)=n} \mathcal{C}_{g,\mu}(a) p_\mu.$$

Because $\mathcal{C}_{g,n}(p; a)$ is a formal power series in p_1, p_2, \dots , for each n , one can obtain from it a formal power series $\Phi_{g,n}(x_1, \dots, x_n; a)$ by applying the following linear symmetrization operator [11, 5]:

$$p_\mu \mapsto (\sqrt{-1})^{-(n+|\mu|)} \delta_{l(\mu),n} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}.$$

Hence by (23) we have

$$\begin{aligned} & \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x_1, \dots, x_n; a) \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{l(\mu^1) + \dots + l(\mu^k) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} q^{a\kappa_{\nu^i}/2} \sqrt{-1}^{|\nu^i|} s_{\nu^i}(q^\rho) \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \\ & \quad \cdot (\sqrt{-1})^{-(n+|\mu^1| + \dots + |\mu^k|)} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}, \end{aligned}$$

where $\mu = (\mu_1, \dots, \mu_n) = \mu^1 \cup \dots \cup \mu^k$. In particular,

$$\begin{aligned} & \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \sqrt{-1}^n \Phi_{g,n}(x, \dots, x; a) \\ (24) \quad &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{l(\mu^1) + \dots + l(\mu^k) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} q^{a\kappa_{\nu^i}/2} s_{\nu^i}(q^\rho) \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \\ & \quad \cdot x^{|\mu^1| + \dots + |\mu^k|}. \end{aligned}$$

2.3. Computation of the partition function.

Proposition 2.1. *For the framed local \mathbb{C}^3 , define*

$$(25) \quad Z = \exp \sum_{n \geq 1} \frac{1}{n!} \sum_{g \geq 0} (-1)^{g-1+n} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x; a).$$

Then one has

$$(26) \quad Z = \sum_{n=0}^{\infty} \frac{e^{-an(n-1)\lambda/2+n\lambda/2}}{\prod_{j=1}^n (1 - e^{j\lambda})} x^n.$$

Proof. By (24) we have

$$\begin{aligned} & \exp \sum_{n \geq 1} \frac{1}{n!} \sum_{g \geq 0} \sqrt{-1}^n \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x; a) \\ &= \exp \left(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{|\mu^1|, \dots, |\mu^k| > 0} \prod_{i=1}^k \sum_{|\nu^i|=|\mu^i|} q^{a\kappa_{\nu^i}/2} s_{\nu^i}(q^\rho) \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} x^{|\mu^i|} \right) \\ &= 1 + \sum_{|\mu| > 0} \sum_{|\nu|=|\mu|} q^{a\kappa_\nu/2} s_\nu(q^\rho) \frac{\chi_\nu(\mu)}{z_\mu} \cdot x^{|\mu|} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu|=n} x^{|\nu|} q^{a\kappa_\nu/2} s_\nu(q^\rho) \sum_{|\mu|=n} \frac{\chi_\nu(\mu)}{z_\mu} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu|=n} x^{|\nu|} q^{a\kappa_\nu/2} s_\nu(q^\rho) \delta_{\nu,(n)} \\ &= 1 + \sum_{n=1}^{\infty} x^n q^{a\kappa_{(n)}/2} s_{(n)}(q^\rho) = \sum_{n=0}^{\infty} \frac{q^{(2a+1)n(n-1)/4}}{[n]!} x^n. \end{aligned}$$

In the last equality we have used the following identity:

$$(27) \quad s_{(n)}(q^\rho) = \frac{q^{n(n-1)/4}}{[n]!}.$$

The proof is completed by changing λ to $\sqrt{-1}\lambda$:

$$Z = \sum_{n=0}^{\infty} \frac{e^{-(2a+1)n(n-1)\lambda/4}}{\prod_{j=1}^n (e^{-j\lambda/2} - e^{j\lambda/2})} x^n = \sum_{n=0}^{\infty} \frac{e^{-an(n-1)\lambda/2+n\lambda/2}}{\prod_{j=1}^n (1 - e^{j\lambda})} x^n.$$

□

2.4. Differential equation satisfied by the partition function.

Write $Z = a_0 + a_1 + \dots$, where

$$(28) \quad a_n = e^{-an(n-1)\lambda/2+n\lambda/2} \frac{x^n}{\prod_{j=1}^n (1 - e^{j\lambda})}.$$

Then one has

$$(29) \quad \frac{a_{n+1}}{a_n} = \frac{e^{\lambda/2}x}{1 - e^{(n+1)\lambda}} e^{-an\lambda},$$

and so

$$(30) \quad (1 - e^{(n+1)\lambda})a_{n+1} - e^{\lambda/2}xe^{-an\lambda}a_n = 0.$$

By summing over n , one gets:

Theorem 2.2. *For the partition function associated with the Hurwitz numbers define above, the following equation is satisfied:*

$$(31) \quad (1 - \hat{y} - e^{\lambda/2}\hat{x}\hat{y}^{-a})Z = 0,$$

where

$$(32) \quad \hat{x} = x, \quad \hat{y} = e^{\lambda x \frac{\partial}{\partial x}}.$$

This proves the Gukov-Sulkowski conjecture [12] for the framed mirror curve of \mathbb{C}^3 . Recall the BKMP conjecture for \mathbb{C}^3 (see [4]) has been proved [5, 20]. More precisely,

$$W_{g,n}(x_1, \dots, x_n) = (-1)^{g-1+n} \partial_{x_1} \cdots \partial_{x_n} \Phi_{g,n}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

satisfies the Eynard-Orantin recursion defined by the following algebraic curve (cf. [20, (8)]):

$$(33) \quad x - y^a + y^{a+1} = 0.$$

Our result shows that one needs to change the above equation to

$$(34) \quad A(x, y) = 1 - y - xy^{-a} = 0$$

before taking the quantization. Further, higher order quantum corrections introduce an extra factor of $e^{\lambda/2}$ for \hat{x} , i.e., one should take $\hat{x} = e^{\lambda/2} \cdot x$.

3. OPEN STRING INVARIANTS OF THE RESOLVED CONIFOLD AND QUANTIZATION OF ITS MIRROR CURVE

3.1. Open string amplitudes of the resolved conifold with one special outer brane. Let us first recall the result in [22] about the open string amplitude for the resolved conifold with one special outer

brane and framing a by the theory of the topological vertex [1, 15], corresponding to the following toric diagrams as follows:

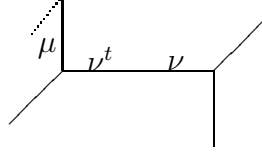


Figure 1.

This is the Figure 1(b) case in [22]. The case of Figure 1(a) can be treated in a similar fashion. We have

$$\tilde{Z}^{(a)}(\lambda; t; \mathbf{p}) = \sum_{\mu, \nu, \eta} q^{a\kappa_\mu/2} C_{\mu, (0), \nu^t}(q) \cdot Q^{|\nu|} \cdot C_{\nu, (0), (0)}(q) \cdot \frac{\chi_\mu(\eta)}{z_\eta \sqrt{-1}^{l(\eta)}} p_\eta.$$

where $q = e^{\sqrt{-1}\lambda}$ and $Q = -e^{-t}$. The normalized open string amplitudes are defined by:

$$\hat{\tilde{Z}}^{(a)}(\lambda; t; \mathbf{p}) = \frac{\tilde{Z}^{(a)}(\lambda; t; \mathbf{p})}{\tilde{Z}^{(a)}(\lambda; t; \mathbf{p})|_{\mathbf{p}=\mathbf{0}}}.$$

Write

$$(35) \quad \hat{\tilde{Z}}^{(a)}(\lambda; t; \mathbf{p}) = \exp \sum_{\mu \in \mathcal{P}^+} \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \tilde{F}_{g;k;\mu}^{(a)} \lambda^{2g-2+l(\mu)} e^{-kt} p_\mu,$$

where \mathcal{P}^+ is the set of nonempty partitions. The following formula is proved in [22] as a mathematical formulation of the full Mariño-Vafa Conjecture [17]:

$$(36) \quad \exp \sum_{\mu \in \mathcal{P}^+} \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \sqrt{-1}^{l(\mu)} \tilde{F}_{g;k;\mu}^{(a)} \lambda^{2g-2+l(\mu)} e^{(|\mu|/2-k)t} p_\mu \\ = \sum_{\mu} s_\mu \cdot q^{a\kappa_\mu/2} \cdot \dim_q R_\mu.$$

(Unfortunately $l(\mu)$ is missing from $\lambda^{2g-2+l(\mu)}$ in [22].) In this formula, $\dim_q R_\mu$ is the quantum dimension that gives the colored large N HOM-FLY polynomials of the unknot are given by the quantum dimension [17, (5.4)]:

$$(37) \quad \dim_q R_\mu = \prod_{1 \leq i < j \leq l(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \cdot \prod_{i=1}^{l(\mu)} \frac{\prod_{j=1}^{\mu_i} [j - i]_{e^t}}{\prod_{j=1}^{\mu_i} [j - i + l(\mu)]}.$$

In [22], it is realized as a specialization of the Schur function as follows:
If for $n \geq 1$ one has

$$(38) \quad p_n(\mathbf{y}) = \frac{e^{nt/2} - e^{-nt/2}}{[n]},$$

then one has:

$$(39) \quad s_\mu(\mathbf{y}) = \dim_q R_\mu = \prod_{x \in \mu} \frac{[c(x)]_{e^t}}{[h(x)]},$$

where

$$(40) \quad [n]_{e^t} = e^{t/2} q^{n/2} - e^{-t/2} q^{-n/2}.$$

In particular,

$$(41) \quad s_{[n]}(\mathbf{y}) = \prod_{j=1}^n \frac{e^{t/2} q^{(j-1)/2} - e^{-t/2} q^{-(j-1)/2}}{q^{j/2} - q^{-j/2}}.$$

3.2. Symmetrization. Taking logarithm on both sides of (36), we get:

$$(42) \quad \begin{aligned} & \sum_{g=0}^{\infty} \sum_{\mu \in \mathcal{P}^+} \sum_{k=0}^{\infty} \sqrt{-1}^{l(\mu)} \tilde{F}_{g;k;\mu}^{(a)} e^{-kt} \lambda^{2g-2+l(\mu)} p_\mu \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{\mu^1 \cup \dots \cup \mu^k = \mu \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{-|\nu^i|t/2} q^{a\kappa_{\nu^i}/2} \dim_q R_{\nu^i} \\ & \quad \cdot \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} p_{\mu^i}. \end{aligned}$$

We apply the following linear symmetrization operator:

$$p_\mu \mapsto (\sqrt{-1})^{-l(\mu)} \sum_{\sigma \in S_{l(\mu)}} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(l(\mu))}^{\mu_{l(\mu)}}$$

to get:

$$\begin{aligned} & \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x_1, \dots, x_n) \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{l(\mu^1) + \dots + l(\mu^k) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{-|\nu^i|t/2} q^{a\kappa_{\nu^i}/2} \dim_q R_{\nu^i} \\ & \quad \cdot \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} \cdot \sum_{\sigma \in S_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}, \end{aligned}$$

where $\mu = (\mu_1, \dots, \mu_n) = \mu^1 \cup \dots \cup \mu^k$. In particular,

$$\begin{aligned}
 (43) \quad & \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x) \\
 &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{l(\mu^1) + \dots + l(\mu^k) = n \\ |\mu^1|, \dots, |\mu^k| > 0}} \prod_{i=1}^k \sum_{|\nu^i| = |\mu^i|} e^{-|\nu^i|t/2} q^{a\kappa_{\nu^i}/2} \dim_q R_{\nu^i} \\
 & \quad \cdot \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} x^{|\mu^i|}.
 \end{aligned}$$

3.3. Computation of the partition function.

Proposition 3.1. *For the framed local \mathbb{C}^3 , define*

$$(44) \quad Z = \exp \sum_{n \geq 1} \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x).$$

Then one has

$$(45) \quad Z = \sum_{n=0}^{\infty} \prod_{j=1}^n \frac{1 - e^{-t} q^{-(j-1)}}{1 - q^{-j}} q^{an(n-1)/2 - n/2} x^n.$$

Proof. By (43) we have

$$\begin{aligned}
 & \exp \sum_{n \geq 1} \frac{1}{n!} \sum_{g \geq 0} \lambda^{2g-2+n} \Phi_{g,n}(x, \dots, x) \\
 &= 1 + \sum_{|\mu| > 0} \sum_{|\nu| = |\mu|} e^{-|\nu|t/2} q^{a\kappa_{\nu}/2} \dim_q R_{\nu} \frac{\chi_{\nu}(\mu)}{z_{\mu}} \cdot x^{|\mu|} \\
 &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu|=n} x^{|\nu|} e^{-|\nu|t/2} q^{a\kappa_{\nu}/2} \dim_q R_{\nu} \sum_{|\mu|=n} \frac{\chi_{\nu}(\mu)}{z_{\mu}} \\
 &= 1 + \sum_{n=1}^{\infty} \sum_{|\nu|=n} x^{|\nu|} e^{-|\nu|t/2} q^{a\kappa_{\nu}/2} \dim_q R_{\nu} \delta_{\nu,(n)} \\
 &= 1 + \sum_{n=1}^{\infty} x^n e^{-nt/2} q^{a\kappa_{(n)}/2} \dim_q R_{(n)} \\
 &= \sum_{n=0}^{\infty} e^{-nt/2} \prod_{j=1}^n \frac{e^{t/2} q^{(j-1)/2} - e^{-t/2} q^{-(j-1)/2}}{q^{j/2} - q^{-j/2}} q^{an(n-1)/2} x^n \\
 &= \sum_{n=0}^{\infty} \prod_{j=1}^n \frac{e^{-t} - q^{j-1}}{1 - q^j} \cdot q^{an(n-1)/2 + n/2} x^n.
 \end{aligned}$$

□

3.4. Differential equation satisfied by the partition function.

Write $Z = a_0 + a_1 + \cdots$, where

$$(46) \quad a_n = \sum_{n=0}^{\infty} \prod_{j=1}^n \frac{e^{-t} - q^{j-1}}{1 - q^j} \cdot q^{an(n-1)/2 + n/2} x^n.$$

Then one has

$$(47) \quad \frac{a_{n+1}}{a_n} = \frac{e^{-t} - q^n}{1 - q^{n+1}} q^{an+1/2} x,$$

and so

$$(48) \quad (1 - q^{n+1})a_{n+1} + xq^{(a+1)n+1/2}a_n - e^{-t}q^{an+1/2}xa_n = 0.$$

By summing over n , one gets:

Theorem 3.2. *For the partition function associated with the Hurwitz numbers define above, the following equation is satisfied:*

$$(49) \quad (1 - \hat{y} + q^{1/2}\hat{x}\hat{y}^{a+1} - q^{1/2}e^{-t}\hat{x}\hat{y}^a)Z = 0,$$

where

$$(50) \quad \hat{x} = x, \quad \hat{y} = e^{-\sqrt{-1}\lambda x \frac{\partial}{\partial x}}.$$

This proves the Gukov-Sulkowski conjecture [12] for the framed mirror curve for the resolved conifold. In [21] it has been proved that by counting the disc invariants one can get the following equation of the framed mirror curve of the resolved conifold with an outer brane and framing a :

$$(51) \quad y + xy^{-a} - 1 - e^{-t}xy^{-a-1} = 0.$$

By changing x to $-x$ and a to $-a-1$, one gets the following equation:

$$(52) \quad A(x, y) = 1 - y + x^{a+1} - e^{-t}xy^a = 0.$$

Our result indicates that when taking the quantization higher order quantum corrections introduce an extra factor of $q^{1/2}$ for \hat{x} , i.e., one should take $\hat{x} = q^{1/2} \cdot x$.

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